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On Combinatorial Properties, Invariants and Structures of the Action of $A_n \times A_n$ on $X \times Y$

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ABSTRACT

This paper investigates the permutation group action of the direct product $A_n \times A_n$ on the Cartesian product $X \times Y$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ are disjoint sets of equal cardinality. Despite the natural symmetry of this action, its combinatorial and structural properties have received little attention in the literature. We prove that the action is transitive and imprimitive for all $n \geq 3$. The rank of the action is shown to be 9 for $n = 3$ and subdegrees $1^{\times 9}$. For all $n \geq 4$, the subdegrees are $1, (n-2)^{\times 2}, (n-1)^2$. Furthermore, we demonstrate that all nontrivial suborbits are self-paired when $n \geq 4$, and we construct the corresponding suborbital graphs, analyzing their regularity, connectedness, and girth.

Notation and preliminary results

Definition 1.1 (Group Action).

Let G be a group and Ω a nonempty set. A **left action** of G on Ω is a mapping $G \times \Omega \to \Omega$, denoted $(g, \omega) \mapsto g\omega$, satisfying $(g_1 g_2)\omega = g_1(g_2\omega)$ and $e\omega = \omega$ for all $g_1, g_2 \in G$, $\omega \in \Omega$, where e is the identity of G. The set Ω is then called a G**-set**. [4]

Definition 1.2 (Orbits and Transitivity).

For a fixed $\omega \in \Omega$, the set $G\omega = \{g\omega \mid g \in G\}$ is the **orbit** of ω. The orbits partition Ω. If there is exactly one orbit, the action is **transitive**.

Definition 1.4 (Stabilizer).

For $\omega \in \Omega$, the **stabilizer** $G_\omega = \{g \in G \mid g\omega = \omega\}$ is a subgroup of G.

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Theorem 1.5 (Orbit–Stabilizer Theorem).

For a finite group G acting on Ω and $\omega \in \Omega$, $| G\omega | = [G: G_\omega]$. [3]

Definition 1.6 (Blocks and Primitivity).

Let G act transitively on the set Ω. A subset $B \subseteq \Omega$ is a **block** if for every $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$. The trivial blocks are $\emptyset$, Ω, and singletons. If no nontrivial blocks exist, the action is **primitive**; otherwise, it is **imprimitive**. [2]

Definition 1.7 (Suborbits and Rank).

Suppose G acts transitively on Ω and fix $\omega \in \Omega$. The orbits of the stabilizer G_ω on Ω are called **suborbits**. The number r of suborbits is the **rank** of G (with respect to this action), and the sizes of the suborbits are the **subdegrees**.

Definition 1.8 (Paired Suborbits)

Let Δ be a suborbit of G_ω. The **paired suborbit** Δ^* is defined by

$$\Delta^* = \{\psi \in \Omega \mid \exists\, g \in G: g\psi = \omega, g\omega \in \Delta\}.$$

If $\Delta^* = \Delta$, then Δ is **self-paired**.

Definition 1.9 (Suborbital Graphs).

Let $O \subseteq \Omega \times \Omega$ be a G-orbit (a **suborbital**). For a fixed $\omega \in \Omega$, the set $\{\psi \in \Omega \mid (\omega, \psi) \in O\}$ is a suborbit. Conversely, each suborbit Δ corresponds to a suborbital $O = \{(g\omega, g\psi) | g \in G, \psi \in \Delta\}$.The **suborbital graph** Γ has vertex set Ω and a directed edge from α to β whenever $(\alpha, \beta) \in O$. If Δ is self-paired, Γ is undirected. [1]

2. Transitivity and Imprimitivity of the Action

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be disjoint sets, and $G = A_n \times A_n$ act on $\Omega = X \times Y$ by

$$(g,h)(x,y) = (gx, hy), (g,h) \in G, (x,y) \in \Omega.$$

Theorem 2.1 (Transitivity).

The action of G on Ω is transitive if and only if $n \geq 3$.

Proof.

Case $n = 2$. In this case A_2 is trivial and so $G \cong \{e\}$. Clearly, $| \Omega | = 4$ but $| G\omega | = 1$ for any $\omega \in \Omega$. Hence the action is intransitive.

Case $n \geq 3$. Fix $(x_1, y_1) \in \Omega$. The stabilizer $H = G_{(x_1,y_1)} = \{(g,h) \in G | gx_1 = x_1, hy_1 = y_1\}$ is isomorphic to $A_{n-1} \times A_{n-1}$. Indeed, the condition $gx_1 = x_1$ forces g to permute the remaining $n-1$ points, and similarly for h. Since A_{n-1} acts transitively on $\{x_2, \dots, x_n\}$ and on $\{y_2, \dots, y_n\}$, we have $| H | = \frac{((n-1)!)^2}{4}$. Now $| G | = \frac{(n!)^2}{4}$ and $| \Omega | = n^2$. By the Orbit–Stabilizer Theorem,

$$|G(x_1, y_1)| = \frac{|G|}{|H|}$$

$$= \frac{\frac{(n!)^2}{4}}{\frac{((n-1)!)^2}{4}}$$

$$= n^2$$

$$= | \Omega |.$$

Thus, the orbit of (x_1, y_1) is the whole set, proving transitivity. ■

Theorem 2.2 (Imprimitivity).

For all $n \geq 3$, the action of G on Ω is imprimitive.

Proof. Consider the subset

$$B = \{(x_1, y_1), (x_1, y_2), \dots, (x_1, y_n)\} \subseteq \Omega.$$

$|B| = n$, which divides $|\Omega| = n^2$. We show B is a nontrivial block. Take any $g = (g_X, g_Y) \in G$.

- If $g_X(x_1) = x_1$, then g maps B to itself: for each y_j, $g(x_1, y_j) = (x_1, g_Y(y_j)) \in B$. Hence $gB = B$.

- If $g_X(x_1) \neq x_1$, say $g_X(x_1) = x_k$ with $k \geq 2$, then every element of gB has first coordinate x_k, whereas every element of B has first coordinate x_1. Thus $gB \cap B = \emptyset$.

Since B is neither a singleton nor the whole set, it is a nontrivial block. Therefore, the action is imprimitive. ∎

3. Rank and Subdegrees

We now determine the rank and subdegrees of the action. The case $n = 3$ is exceptional and treated separately.

Lemma 3.1 (The Case $n = 3$).

For $n = 3$, the stabilizer $H = G_{(x_1, y_1)}$ is trivial. Consequently, the rank is

$|\Omega| = 9$ and each subdegree is 1.

Proof. For $n = 3$, A_2 is trivial, so $A_{n-1} = A_2 \cong \{e\}$. Hence $H \cong \{e\} \times \{e\}$. The orbits of H on Ω are therefore singletons. There are exactly 9 such orbits, each of size 1. ∎

Theorem 3.2 (Rank and Subdegrees for $n \geq 4$).

For all $n \geq 4$, the action of G on Ω has rank 4 and subdegrees

$$1, (n-1)^{\times 2}, (n-1)^2.$$

Proof. Fix $(x_1, y_1) \in \Omega$ and let $H = G_{(x_1, y_1)} \cong A_{n-1} \times A_{n-1}$. The action of H on Ω partitions Ω into the following four disjoint orbits:

$\Delta_0 = \{(x_1, y_1)\}$, of size 1.

$\Delta_1 = \{(x_1, y_j) \mid 2 \leq j \leq n\}$, of size $n-1$. This is an orbit because H fixes x_1 and acts as A_{n-1} on the y-coordinates.

$\Delta_2 = \{(x_i, y_1) \mid 2 \leq i \leq n\}$ of size $n-1$. This is an orbit because H fixes y_1 and acts as A_{n-1} on the x-coordinates.

$\Delta_3 = \{(x_i, y_j) \mid 2 \leq i, j \leq n\}$ of size $(n-1)^2$. This is an orbit because A_{n-1} acts transitively on the remaining x-coordinates and independently on the remaining y-coordinates.

To prove that these are the only suborbits of G, it suffices to show that $P = \{\Delta_0, \Delta_1, \Delta_2, \Delta_3\}$ is a partition of Ω.

Clearly, $\Delta_i \neq \emptyset$ for each $i = 0,1,2,3$ and $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j (i, j = 0,1,2,3)$.

Now,

$$\begin{aligned} \sum_{i=0}^{3} |\Delta_i| &= 1 + 2(n-1) + (n-1)^2 \\ &= 1 + (n-1)\big(2 + (n-1)\big) \\ &= 1 + (n-1)(n+1) \\ &= n^2 \\ &= |\Omega| \end{aligned}$$

and it follows that P is a partition of Ω. ∎

4. Pairing of Suborbits

Theorem 4.1.

The suborbits of $G = A_3 \times A_3$ on Ω are paired.

Proof. The action of $G = A_3 \times A_3$ on $X \times Y$ has eight nontrivial suborbits, which are labeled $\Delta_1, \Delta_2, \ldots, \Delta_8$.

Since the order of G is even, the action must have at least one suborbit that is self-paired.

First, take the point (x_1, y_2) in Δ_1 and let $g = (e_X, (y_1 y_3 y_2))$ be an element of G. Then applying g to (x_1, y_2) gives (x_1, y_1), and applying g to (x_1, y_1) gives (x_1, y_3), which lies in Δ_2. Therefore, $\Delta_1^* = \Delta_2$; in other words, Δ_1 is paired with Δ_2.

Next, take the point (x_2, y_1) in Δ_3 and let $g = ((x_1 x_3 x_2), e_Y) \in G$. Then $g(x_2, y_1) = (x_1, y_1)$, and then $g(x_1, y_1) = (x_3, y_1)$, which lies in Δ_6. Hence, $\Delta_3^* = \Delta_6$; that is, Δ_3 is paired with Δ_6.

Now take the point (x_2, y_2) in Δ_4. Suppose $g = ((x_1 x_3 x_2), (y_1 y_3 y_2)) \in G$. Then $g(x_2, y_2) = (x_1, y_1)$ and $g(x_1, y_1) = (x_3, y_3)$, which lies in Δ_8. So $\Delta_4^* = \Delta_8$.

Finally, take the point (x_2, y_3) in Δ_5 and let $g = ((x_1 x_3 x_2), (y_1 y_2 y_3)) \in G$. Then $g(x_2, y_3) = (x_1, y_1)$ and $g(x_1, y_1) = (x_3, y_2)$, which lies in Δ_7. Therefore, $\Delta_5^* = \Delta_7$. ■

Theorem 4.2. The suborbits of $G = A_n \times A_n$ on $X \times Y$ are self-paired for all $n \geq 4$.

Proof. Let $n \geq 4$. From Theorem 3.2, then G has 3 non-trivial sub orbits, namely Δ_1, Δ_2 and Δ_3. Consider $(x_1, y_2) \in \Delta_1$ and $g = (e_X, (y_1 y_3 y_2)) \in G$. Then $g(x_1, y_2) = (x_1, y_1)$ and hence $g(x_1, y_1) = (x_1, y_3) \in \Delta_1$. So, $\Delta_1^* = \Delta_1$; i.e., Δ_1 is self-paired. Next, take $(x_2, y_1) \in \Delta_2$ and let $g = ((x_1 x_3 x_2), e_Y) \in G$. Then $g(x_2, y_1) = (x_1, y_1)$ and hence $g(x_1, y_1) = (x_3, y_1) \in \Delta_2$. So, $\Delta_2^* = \Delta_2$; i.e., Δ_2 is self-paired. Finally, consider $(x_2, y_2) \in \Delta_3$. Let $g = ((x_1 x_3 x_2), (y_1 y_3 y_2)) \in G$.
Then $g(x_2, y_2) = (x_1, y_1)$ and therefore $g(x_1, y_1) = (x_3, y_3) \in \Delta_3$. Therefore, $\Delta_3^* = \Delta_3$; i.e., Δ_3 is self-paired. Therefore, all the suborbits of the action are self-paired. ■

5. Suborbital Graphs

Theorem 5.1. The suborbital graphs of $A_n \times A_n$ on $X \times Y$ are undirected for all $n \geq 4$.

Proof. Each of the suborbits is self-paired, from Theorem 4.2 and is therefore undirected by Definition 1.9. ■

A suborbital graph of the action has $X \times Y$ as its vertex set.

Now, the construction and properties of the three non-trivial graphs of the action are as follows:

(i) The suborbital O_1 corresponding to the suborbit Δ_1 is

$$O_1 = \{((gX, gY)(x_1, y_1), (gX, gY)(x_1, y_2)) \mid (gX, gY) \in G, (x_1, y_2) \in \Delta_1\}.$$

The suborbital graph Γ_1 corresponding to the suborbital O_1 has an edge from vertex (u, v) to vertex (x, y) if and only if $u = x$ and $v \neq y$.

The graph is regular of degree $(n-1)$ since vertex (u, v) is connected to $(n-1)$ vertices (u, w) where $v \neq w$. It is disconnected since there is no path for vertex (u, v) to vertex (w, y) if $u \neq w$ and $v \neq y$ and has n connected components since each vertex in the graph is connected to $(n-1)$ vertices for there is an edge from vertex (u, v) to vertex (x, y) if and only if $u = x$ and $v \neq y$ hence there are n connected components. Now, the number of connected components is $\frac{|X \times Y|}{n} = \frac{n^2}{n} = n$. It has girth 3 since $(x_1, y_1), (x_1, y_2), (x_1, y_3)$ form a cycle in Γ_1.

(ii) The suborbital O_2 corresponding to the suborbit $\triangle_2$ is

$$O_2 = \{((gX, gY)(x_1, y_1), (gX, gY)(x_2, y_1)) \mid (gX, gY) \in G, (x_2, y_1) \in \Delta_2\}.$$

The suborbital graph Γ_2 corresponding to the suborbital O_2 has an edge from vertex (u, v) to vertex (x, y) if and only if $u \neq x$ and $v = y$.

This graph is isomorphic to Γ_1 and therefore, the two have the same properties.

(iii) The suborbital O_3 corresponding to the suborbit $\triangle_3$ is

$$O_3 = \{((gX, gY)(x_1, y_1), (gX, gY)(x_2, y_2)) \mid (gX, gY) \in G, (x_2, y_2) \in \Delta_3\}.$$

The suborbital graph Γ_3 corresponding to the suborbital O_3 has an edge from vertex (u, v) to vertex (x, y) if and only if $\{u, v\} \cap \{x, y\} = \emptyset$.

The graph is regular of degree $(n - 1)^2$. The graph is connected since there is a path between every two vertices. It has girth 3 since the vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ form a cycle in Γ_3.

6. Conclusion

In this paper, we systematically analyzed the permutation group action of $A_n \times A_n$ on $X \times Y$, where $|X| = |Y| = n$. We established the following key results:

1. Transitivity and Imprimitivity: The action is transitive for all $n \geq 3$ and imprimitive for all $n \geq 3$ with an explicit block system given by fixing the first coordinate.

2. Rank and Subdegrees: For $n = 3$, the action has rank 9 with all subdegrees $1^{\times 9}$. For $n \geq 4$, the rank is 4 with subdegrees $1, (n-1)^{\times 2}$ and $(n-1)^2$.

3. Pairing of Suborbits: For $n = 3$, suborbits occur in nontrivial pairs; for $n \geq 4$, every suborbit is self-paired, a rare and highly symmetric property.

4. Suborbital Graphs: For $n \geq 4$, all suborbital graphs are undirected. We explicitly described three families: two are disjoint unions of n complete graphs K_n (girth 3, regular of degree $n-1$, and the third is a connected regular graph of degree $(n-1)^2$ with girth 3.

These results significantly extend the known combinatorial theory of direct product actions and provide a complete invariant set for this family of permutation groups.

Recommendations for Further Research

Having achieved the objectives of the present study, several interesting avenues remain unexplored. In particular, one may investigate the transitivity, primitivity, ranks, subdegrees, and suborbital graphs associated with the following actions:

1. The action of the Cartesian product $S_n \times S_n$ on either $X^{(r)} \times X^{(r)}$ or $X^{[r]} \times X^{[r]}$, where S_n is the symmetric group on n symbols.

2. The action of the Cartesian product $A_n \times A_n$ on either $X^{(r)} \times X^{(r)}$ or $X^{[r]} \times X^{[r]}$, where A_n is the alternating group on n symbols.

Here, $X = \{1,2, \dots, n\}$, $X^{(r)}$ denotes the set of all r-element subsets of X, and $X^{[r]}$ denotes the set of all ordered r-tuples of distinct elements from X.

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